Solutions to Review Problems for Exam 1

1. There are 5 red chips and 3 blue chips in a bowl. The red chips are numbered 1, 2, 3, 4, 5 respectively, and the blue chips are numbered 1, 2, 3 respectively. If two chips are to be drawn at random and without replacement, find the probability that these chips are have either the same number or the same color.

Solution: Let \( R \) denote the event that the two chips are red. Then the assumption that the chips are drawn at random and without replacement implies that

\[
\Pr(R) = \binom{5}{2} / \binom{8}{2} = \frac{5}{14}.
\]

Similarly, if \( B \) denotes the event that both chips are blue, then

\[
\Pr(B) = \binom{3}{2} / \binom{8}{2} = \frac{3}{28}.
\]

It then follows that the probability that both chips are of the same color is

\[
\Pr(R \cup B) = \Pr(R) + \Pr(B) = \frac{13}{28},
\]

since \( R \) and \( B \) are mutually exclusive.

Let \( N \) denote the event that both chips show the same number. Then,

\[
\Pr(N) = \binom{3}{2} / \binom{8}{2} = \frac{3}{28}.
\]

Finally, since \( R \cup B \) and \( N \) are mutually exclusive, then the probability that the chips are have either the same number or the same color is

\[
\Pr(R \cup B \cup N) = \Pr(R \cup B) + \Pr(N) = \frac{13}{28} + \frac{3}{28} = \frac{16}{28} = \frac{2}{7}.
\]
2. A person has purchased 10 of 1,000 tickets sold in a certain raffle. To determine the five prize winners, 5 tickets are drawn at random and without replacement. Compute the probability that this person will win at least one prize.

**Solution:** Let \( N \) denote the event that the person will not win any prize. Then

\[
\Pr(N) = \binom{995}{10} \binom{1000}{10}; \tag{1}
\]

that is, the probability of purchasing 10 non-winning tickets. It follows from (1) that

\[
\Pr(N) = \frac{(990)(989)(988)(987)(986)}{(1000)(999)(998)(997)(996)} = \frac{435841667261}{458349513900} \approx 0.9509. \tag{2}
\]

Thus, using the result in (2), the probability of the person winning at least one of the prizes is

\[
\Pr(N^c) = 1 - \Pr(N) \approx 1 - 0.9509 = 0.0491,
\]

or about 4.91\%.

□

3. Let \((\mathcal{C}, \mathcal{B}, \Pr)\) denote a probability space, and let \( E_1, E_2 \) and \( E_3 \) be mutually disjoint events in \( \mathcal{B} \). Find \( \Pr[(E_1 \cup E_2) \cap E_3] \) and \( \Pr(E_1^c \cup E_2^c) \).

**Solution:** Since \( E_1, E_2 \) and \( E_3 \) are mutually disjoint events, it follows that \((E_1 \cup E_2) \cap E_3 = \emptyset\); so that

\[
\Pr[(E_1 \cup E_2) \cap E_3] = 0.
\]
Next, use De Morgan’s law to compute
\[
\Pr(E_1^c \cup E_2^c) = \Pr([E_1 \cap E_2]^c)
\]
\[
= \Pr(\emptyset^c)
\]
\[
= \Pr(\mathcal{C})
\]
\[
= 1.
\]
\[\square\]

4. Let \((\mathcal{C}, \mathcal{B}, \Pr)\) denote a probability space, and let \(A\) and \(B\) events in \(\mathcal{B}\). Show that
\[
\Pr(A \cap B) \leq \Pr(A) \leq \Pr(A \cup B) \leq \Pr(A) + \Pr(B).
\] (3)

**Solution:** Since \(A \cap B \subseteq A\), it follows that
\[
\Pr(A \cap B) \leq \Pr(A).
\] (4)

Similarly, since \(A \subseteq A \cup B\), we get that
\[
\Pr(A) \leq \Pr(A \cup B).
\] (5)

Next, use the identity
\[
\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B),
\]
and fact that that
\[
\Pr(A \cap B) \geq 0,
\]
to obtain that
\[
\Pr(A \cup B) \leq \Pr(A) + \Pr(B).
\] (6)

Finally, combine (4), (5) and (6) to obtain (3). \(\square\)

5. Let \((\mathcal{C}, \mathcal{B}, \Pr)\) denote a probability space, and let \(E_1, E_2\) and \(E_3\) be mutually independent events in \(\mathcal{B}\) with probabilities \(\frac{1}{2}, \frac{1}{3}\) and \(\frac{1}{4}\), respectively. Compute the exact value of \(\Pr(E_1 \cup E_2 \cup E_3)\).
Solution: First, use De Morgan’s law to compute
\[
\Pr[(E_1 \cup E_2 \cup E_3)^c] = \Pr(E_1^c \cap E_2^c \cap E_3^c) \quad (7)
\]
Then, since \(E_1, E_2\) and \(E_3\) are mutually independent events, it follows from (7) that
\[
\Pr[(E_1 \cup E_2 \cup E_3)^c] = \Pr(E_1^c) \cdot \Pr(E_2^c) \cdot \Pr(E_3^c),
\]
so that
\[
\Pr[(E_1 \cup E_2 \cup E_3)^c] = (1 - \Pr(E_1))(1 - \Pr(E_2))(1 - \Pr(E_3)) = \left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) \left(1 - \frac{1}{4}\right) = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4},
\]
so that
\[
\Pr[(E_1 \cup E_2 \cup E_3)^c] = \frac{1}{4}. \quad (8)
\]
It then follows from (8) that
\[
\Pr(E_1 \cup E_2 \cup E_3) = 1 - \Pr[(E_1 \cup E_2 \cup E_3)^c] = \frac{3}{4}.
\]
□

6. Let \((\mathcal{C}, \mathcal{B}, \Pr)\) denote a probability space, and let \(E_1, E_2\) and \(E_3\) be mutually independent events in \(\mathcal{B}\) with \(\Pr(E_1) = \Pr(E_2) = \Pr(E_3) = \frac{1}{4}\). Compute \(\Pr((E_1^c \cap E_2^c) \cup E_3^c)\).

Solution: First, use De Morgan’s law to compute
\[
\Pr[((E_1^c \cap E_2^c) \cup E_3^c)^c] = \Pr[(E_1^c \cap E_2^c)^c \cap E_3^c] \quad (9)
\]
Next, use the assumption that \(E_1, E_2\) and \(E_3\) are mutually independent events to obtain from (9) that
\[
\Pr[((E_1^c \cap E_2^c) \cup E_3^c)^c] = \Pr[(E_1^c \cap E_2^c)^c] \cdot \Pr(E_3^c), \quad (10)
\]
where
\[ \Pr[E_3^c] = 1 - \Pr(E_3) = \frac{3}{4}, \] (11)
and
\[ \Pr[(E_1^c \cap E_2^c)^c] = 1 - \Pr[E_1^c \cap E_2^c] \]
\[ = 1 - \Pr[E_1^c] \cdot \Pr[E_2^c], \] (12)
by the independence of \(E_1\) and \(E_2\).
It follows from the calculations in (13) that
\[ \Pr[(E_1^c \cap E_2^c)^c] = 1 - (1 - \Pr[E_1])(1 - \Pr[E_2]) \]
\[ = 1 - \left(1 - \frac{1}{4}\right) \left(1 - \frac{1}{4}\right) \]
\[ = 1 - \frac{3}{4} \cdot \frac{3}{4} \]
\[ = \frac{7}{16} \] (13)
Substitute (11) and the result of the calculations in (13) into (10) to obtain
\[ \Pr[((E_1^c \cap E_2^c) \cup E_3)^c] = \frac{7}{16} \cdot \frac{3}{4} = \frac{21}{64}. \] (14)
Finally, use the result in (14) to compute
\[ \Pr[(E_1^c \cap E_2^c) \cup E_3^c] = 1 - \Pr[((E_1^c \cap E_2^c) \cup E_3)^c] \]
\[ = 1 - \frac{21}{64} \]
\[ = \frac{43}{64}. \]
\[ \square \]
7. A bowl contains 10 chips of the same size and shape. One and only one of these chips is red. Draw chips from the bowl at random, one at a time and without replacement, until the red chip is drawn. Let \(X\) denote the number of draws needed to get the red chip.
(a) Find the pmf of $X$.

**Solution:** Compute

\[
\begin{align*}
\Pr(X = 1) &= \frac{1}{10} \\
\Pr(X = 2) &= \frac{9}{10} \cdot \frac{1}{9} = \frac{1}{10} \\
\Pr(X = 3) &= \frac{9}{10} \cdot \frac{8}{9} \cdot \frac{1}{8} = \frac{1}{10} \\
\vdots \\
\Pr(X = 10) &= \frac{1}{10}
\end{align*}
\]

Thus,

\[
p_X(k) = \begin{cases} 
\frac{1}{10} & \text{for } k = 1, 2, \ldots, 10; \\
0 & \text{elsewhere.}
\end{cases} \tag{15}
\]

(b) Compute $\Pr(X \leq 4)$.

**Solution:** Use (15) to compute

\[
\Pr(X \leq 4) = \sum_{k=1}^{4} p_X(k) = \frac{4}{10} = \frac{2}{5}.
\]

8. Let $X$ have pmf given by $p_X(x) = \frac{1}{3}$ for $x = 1, 2, 3$ and $p(x) = 0$ elsewhere. Give the pmf of $Y = 2X + 1$.

**Solution:** Note that the possible values for $Y$ are 3, 5 and 7

Compute

\[
\Pr(Y = 3) = \Pr(2X + 1 = 3) = \Pr(X = 1) = \frac{1}{3}.
\]
Similarly, we get that
\[ \Pr(Y = 5) = \Pr(X = 2) = \frac{1}{3}, \]
and
\[ \Pr(Y = 7) = \Pr(X = 3) = \frac{1}{3}. \]
Thus,
\[ p_Y(k) = \begin{cases} 
\frac{1}{3} & \text{for } k = 3, 5, 7; \\
0 & \text{elsewhere.}
\end{cases} \]
\[ \square \]

9. Let \( X \) have pmf given by \( p_X(x) = \left(\frac{1}{2}\right)^x \) for \( x = 1, 2, 3, \ldots \) and \( p_X(x) = 0 \) elsewhere. Give the pmf of \( Y = X^3 \).

**Solution:** Compute, for \( y = k^3 \), for \( k = 1, 2, 3, \ldots \),
\[ \Pr(Y = y) = \Pr(X^3 = k^3) = \Pr(X = k) = \left(\frac{1}{2}\right)^k, \]
so that
\[ \Pr(Y = y) = \left(\frac{1}{2}\right)^{y^{1/3}}, \text{ for } y = k^3, \text{ for some } k = 1, 2, 3, \ldots \]
Thus,
\[ p_Y(y) = \begin{cases} 
\left(\frac{1}{2}\right)^{y^{1/3}} & \text{for } y = k^3, \text{ for some } k = 1, 2, 3, \ldots; \\
0 & \text{elsewhere.}
\end{cases} \]
\[ \square \]
10. Let \( f_X(x) = \begin{cases} \frac{1}{x^2} & \text{if } 1 < x < \infty; \\ 0 & \text{if } x \leq 1, \end{cases} \) be the pdf of a random variable \( X \). If \( E_1 \) denote the interval \((1, 2)\) and \( E_2 \) the interval \((4, 5)\), compute \( \Pr(E_1) \), \( \Pr(E_2) \), \( \Pr(E_1 \cup E_2) \) and \( \Pr(E_1 \cap E_2) \).

\[ \text{Solution: Compute} \]

\[ \Pr(E_1) = \int_1^2 \frac{1}{x^2} \, dx = -\left. \frac{1}{x} \right|_1^2 = \frac{1}{2}, \]

\[ \Pr(E_2) = \int_4^5 \frac{1}{x^2} \, dx = -\left. \frac{1}{x} \right|_4^5 = \frac{1}{20}, \]

\[ \Pr(E_1 \cup E_2) = \Pr(E_1) + \Pr(E_2) = \frac{11}{20}, \]

since \( E_1 \) and \( E_2 \) are mutually exclusive, and

\[ \Pr(E_1 \cap E_2) = 0, \]

since \( E_1 \) and \( E_2 \) are mutually exclusive. \( \square \)

11. A mode of a distribution of a random variable \( X \) is a value of \( x \) that maximizes the pdf or the pmf. If there is only one such value, it is called the mode of the distribution. Find the mode for each of the following distributions:

(a) \( p(x) = \left( \frac{1}{2} \right)^x \) for \( x = 1, 2, 3, \ldots \), and \( p(x) = 0 \) elsewhere.

\[ \text{Solution: Note that } p(x) \text{ is decreasing; so, } p(x) \text{ is maximized when } x = 1. \text{ Thus, 1 is the mode of the distribution of } X. \ \square \]

(b) \( f(x) = \begin{cases} 12x^2(1-x), & \text{if } 0 < x < 1; \\ 0 & \text{elsewhere}. \end{cases} \)

\[ \text{Solution: Maximize the function } f \text{ over } [0, 1]. \]

Compute

\[ f'(x) = 24x(1-x) - 12x^2 = 12x(2-3x), \]
so that \( f \) has a critical points at \( x = 0 \) and \( x = \frac{2}{3} \). Since \( f(0) = f(1) = 0 \) and \( f(2/3) > 0 \), it follows that \( f \) takes on its maximum value on \( [0, 1] \) at \( x = \frac{2}{3} \). Thus, the mode of the distribution of \( X \) is \( x = \frac{2}{3} \). □

12. Let \( X \) have pdf \( f_X(x) = \begin{cases} 2x, & \text{if } 0 < x < 1; \\ 0, & \text{elsewhere}. \end{cases} \)

Compute the probability that \( X \) is at least 3/4, given that \( X \) is at least 1/2.

**Solution:** We are asked to compute

\[
\Pr(X \geq \frac{3}{4} \mid X \geq \frac{1}{2}) = \frac{\Pr[(X \geq \frac{3}{4}) \cap (X \geq \frac{1}{2})]}{\Pr(X \geq \frac{1}{2})},
\]

(16)

where

\[
\Pr(X \geq \frac{1}{2}) = \int_{1/2}^{1} 2x \, dx
\]

\[
= x^2 \bigg|_{1/2}^{1} = 1 - \frac{1}{4},
\]

so that

\[
\Pr(X \geq \frac{1}{2}) = \frac{3}{4};
\]

(17)

and

\[
\Pr[(X \geq \frac{3}{4}) \cap (X \geq \frac{1}{2})] = \Pr(X \geq \frac{3}{4})
\]

\[
= \int_{3/4}^{1} 2x \, dx
\]

\[
= x^2 \bigg|_{3/4}^{1} = 1 - \frac{9}{16},
\]
so that

\[ \Pr[(X \geq 3/4) \cap (X \geq 1/2)] = \frac{7}{16}. \]  \hspace{1cm} (18)

Substituting (18) and (17) into (16) then yields

\[ \Pr(X \geq 3/4 \mid X \geq 1/2) = \frac{7}{16} \cdot \frac{3}{4} = \frac{7}{12}. \]

\[ \square \]

13. Divide a segment at random into two parts. Find the probability that the largest segment is at least three times the shorter.

**Solution:** Assume the segment is the interval (0,1) and let \( X \sim \text{Uniform}(0,1) \). Then \( X \) models a random point in (0,1). We have two possibilities: Either \( X \leq 1 - X \) or \( X > 1 - X \); or, equivalently, \( X \leq \frac{1}{2} \) or \( X > \frac{1}{2} \).

Define the events

\[ E_1 = \left( X \leq \frac{1}{2} \right) \quad \text{and} \quad E_2 = \left( X > \frac{1}{2} \right). \]

Observe that \( \Pr(E_1) = \frac{1}{2} \) and \( \Pr(E_2) = \frac{1}{2} \).

The probability that the largest segment is at least three times the shorter is given by

\[ \Pr(E_1)\Pr(1 - X > 3X \mid E_1) + \Pr(E_2)\Pr(X > 3(1 - X) \mid E_2), \]

by the Law of Total Probability, where

\[ \Pr(1 - X > 3X \mid E_1) = \frac{\Pr[(X < 1/4) \cap E_1]}{\Pr(E_1)} = \frac{1/4}{1/2} = \frac{1}{2}. \]

Similarly,

\[ \Pr(X > 3(1 - X) \mid E_2) = \frac{\Pr[(X > 3/4) \cap E_1]}{\Pr(E_2)} = \frac{1/4}{1/2} = \frac{1}{2}. \]

Thus, the probability that the largest segment is at least three times the shorter is

\[ \Pr(E_1)\Pr(1 - X > 3X \mid E_1) + \Pr(E_2)\Pr(X > 3(1 - X) \mid E_2) = \frac{1}{2}. \]

\[ \square \]
14. Let $X$ have pdf $f_X(x) = \begin{cases} \frac{x^2}{9}, & \text{if } 0 < x < 3; \\ 0, & \text{elsewhere.} \end{cases}$

Find the pdf of $Y = X^3$.

**Solution:** First, compute the cdf of $Y$

$$F_Y(y) = \Pr(Y \leq y). \quad (19)$$

Observe that, since $Y = X^3$ and the possible values of $X$ range from 0 to 3, the values of $Y$ will range from 0 to 27. Thus, in the calculations that follow, we will assume that $0 < y < 27$.

From (19) we get that

$$F_Y(y) = \Pr(X^3 \leq y)$$
$$= \Pr(X \leq y^{1/3})$$
$$= F_X(y^{1/3})$$

Thus, for $0 < y < 27$, we have that

$$f_Y(y) = f_X(y^{1/3}) \cdot \frac{1}{3} y^{-3/2}, \quad (20)$$

where we have applied the Chain Rule. It follows from (20) and the definition of $f_X$ that

$$f_Y(y) = \frac{1}{9} [y^{1/3}]^2 \cdot \frac{1}{3} y^{-3/2} = \frac{1}{27}, \quad \text{for } 0 < y < 27. \quad (21)$$

Combining (21) and the definition of $f_X$ we obtain the pdf for $Y$:

$$f_Y(y) = \begin{cases} \frac{1}{27}, & \text{for } 0 < y < 27; \\ 0 & \text{elsewhere;} \end{cases}$$

in other words $Y \sim \text{Uniform}(0, 27)$. \qed