Mathematical Modeling for Environmental Analysis
Chapter 2: Dimensional Analysis and Scaling

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Contents

1 Dimensional Analysis and Scaling 3
   1.1 Dimensional Considerations of Modeling . . . . . . . . . . . 3
       1.1.1 Dimensions in the Logistic Model . . . . . . . . . . . 4
       1.1.2 Exercises . . . . . . . . . . . . . . . . . . . . . . . 6
   1.2 Scaling to Obtain Dimension-free Models . . . . . . . . . . 6
       1.2.1 First order ODE . . . . . . . . . . . . . . . . . . . . 7
       1.2.2 Second Order ODE . . . . . . . . . . . . . . . . . . . 8
       1.2.3 Exercises . . . . . . . . . . . . . . . . . . . . . . . 9

References 15
2 Contents
1
Dimensional Analysis and Scaling

1.1 Dimensional Considerations of Modeling

It seems fairly obvious that any well-posed model ought to be internally consistent in not trying to add a length to a time, nor equate a mass to a velocity, etc. We call such requirements dimensional consistency. As used here, dimension refers to that aspect of a quantity that its various units of measure have in common. For example, if a quantity \( X \) is measured in units of meters, feet, light-years, etc., its dimension is length and we write \([X] = L\), whereas if the quantity \( t \) is measured in units of nanoseconds, days, years, etc., its dimension is time and we write \([t] = T\). Other common dimensions are mass, \( M \), energy, \( E \), and force, \( F \). Notice that \( E = FL \). Dimension here is a very different notion from the notion of dimension which we study in linear algebra. However, there are some considerations that are similar to others in linear algebra. For example, in any given modeling project we like to choose a minimal set of dimensions to use, similarly to our choice in linear algebra of a minimal linearly independent set for a basis. Also, just as we choose a basis in linear algebra from many possibilities, we frequently choose a set of fundamental dimensions for a model that is convenient for a particular problem. Thus, we could equivalently use energy with length, or force with length, since \( F = EL^{-1} \). An analogous situation in linear algebra would be choosing between \( \{(1,0),(0,1)\} \) and \( \{(1,-1),(0,1)\} \) as a basis for the plane. This analogy has lots more to offer, but we won’t pursue it formally.
There are some basic considerations that we take as axiomatic. For example, quantities must have the same dimension in order to be added together. (“You can’t add apples to oranges.”) We wouldn’t try to add a length to a time, or to a velocity. By the same token, the dimension of a sum is the common dimension of the summands. The dimension of a product is the product of the dimensions, so if \([X] = \text{L}\) then \([X^n] = \text{L}^n\). Similarly, the dimension of a quotient is the quotient of the dimensions. Thus, for example, the dimension of a scalar velocity is equal to \(\text{L}/\text{T}\). For this same reason, if both \(X\) and \(Y\) are lengths, \([X] = [Y] = \text{L}\), then their quotient, \(X/Y\), is said to be dimensionless, or dimension-free, and we write

\[
\frac{X}{Y} = \frac{[X]}{[Y]} = 1.
\]

An important example is the dimension of an angle \(\theta\). Since the radian measure of an angle between two line segments intersecting at a point \(P\) is the ratio of the length of a subtended arc of a circle centered at \(P\) to the length of a radius, we must have

\[
[\theta] = \frac{\text{L}}{\text{L}} = 1
\]

for the dimension of \(\theta\). Since we find in many texts that radian measure for an angle is a length of arc along a unit circle, a common (but incorrect) assumption is that the dimension of an angle is length. However, the qualification, “unit circle”, makes that specification of measure equivalent to the ratio in other circles.

### 1.1.1 Dimensions in the Logistic Model

Consider the logistic model for population growth from Chapter ??

\[
\frac{dp}{dt} = rp \left(1 - \frac{p}{K}\right). \tag{1.1}
\]

Although we frequently think casually of a population in terms of “number of individuals,” we can benefit from a bit more sophistication in dimensional consideration. Thinking of interacting populations, especially predator-prey interactions in which one populations subsists primarily by eating the other, it becomes helpful to measure both in terms of mass (frequently called “biomass” because of its nature). This point of view facilitates thinking of the interaction ecologically as one of the many flows of mass and energy in our environment. We will always adopt this point of view, writing \([p] = \text{M}\).

Starting on the left side of the differential equation (1.1), we assert that

\[
\frac{dp}{dt} \sim \lim_{\Delta t \to 0} \frac{\Delta p}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta p}{\Delta t} = \lim_{\Delta t \to 0} \frac{[\Delta p]}{[\Delta t]} = \lim_{\Delta t \to 0} \frac{\text{M}}{\text{T}} = \frac{\text{M}}{\text{T}}. \tag{1.2}
\]
Although you probably easily agreed that the dimension of a velocity is $LT^{-1}$, you may have found a step or two of this argument to have somewhat dubious rigor. A proof would indeed require more careful argument. Now look at the factor $(1 - p/K)$. The "1" and $(p/K)$ must have the same dimension to be added together (for dimensional consistency). The simplest and easiest way of effecting this, but by no means the only one, is to understand $K$ as a "carrying capacity," measured in the same units, and therefore in the same dimension, as $p$, so that $[p/K] = M/M = 1 = [1]$, and the ratio is a dimensionless proportion. So the "1" in $(1 - p/K)$ is also dimensionless, or a "pure number". Finally we consider $[r]$, which must be determined so that the two sides of the equation are dimensionally consistent, i.e., so that

$$\begin{align*}
\left[ \frac{dp}{dt} \right] &= \left[ r p \left( 1 - \frac{p}{K} \right) \right] \\
\text{or} \\
\frac{M}{T} &= [r] \cdot [p] \cdot \left[ 1 - \frac{p}{K} \right] = [r] \cdot M \cdot 1
\end{align*}$$

and thus

$$[r] = \frac{1}{T}.$$

This is easier to understand by thinking about the equivalent statement,

$$\left[ \frac{1}{r} \right] = \text{T},$$

that is, $1/r$ is a time.

One of the fun activities in mathematical modeling is the interpretation of various parts of a model, for example, figuring out what is the significance of the time $1/r$ for the logistic model above. If $p(t)$ is a solution of this model with $p(0)$ very close to, but not equal to, 0, then $p(t) \approx 0$ for quite awhile. During such times, $p(t)/K \approx 0$ also, so that

$$\frac{dp}{dt} \approx rp. \quad (1.3)$$

It is frequently (and fairly easily) shown in calculus classes that functions satisfying $dp/dt = rp$ have a doubling time of $\frac{1}{r} \ln(2)$. Since $dp/dt \approx rp$ when $p$ is small, we see that even logistically growing populations double in about the time $\frac{1}{r} \ln(2)$ while they are small.

Notice that since $1/r$ and $\frac{1}{r} \ln(2)$ are both times, $\ln(2)$ is dimensionless. So, at least in this case, the output of $\ln$ must be dimensionless. How fortunate for the natural log function that it doesn’t have to figure out when to be a time, and when to be something else! It turns out that we usually want both the inputs and outputs of such functions to be dimensionless.

For a more complete, but accessible treatment of dimensional analysis, see David Logan’s text ([Logan1997]).
1.1.2 Exercises

1. If $\pi$ is defined to be the ratio of the circumference of a circle to its diameter, what must be $[\pi]$?

2. In a fashion similar to that used in Equation (1.2), show that if $v(t)$ is a velocity then $[v(t)] = LT^{-1}$.

3. Integrals, as well as derivatives, are defined by limits, and so need considerations similar to those used in Equation (1.2). Beginning with

$$\int_a^b v(t) \, dt = \lim_{n \to \infty} \sum_{i=1}^n v(t_i) \Delta t$$

complete the argument to show that $\left[ \int_a^b v(t) \, dt \right] = L$.

4. Suppose that $v(y)$ is a function with $[v] = V, [y] = Y$ for some dimensions $V, Y$.

(a) Show that

$$\frac{dv}{dy} = \frac{V}{Y}.$$  

(b) Show how the result in part (a.) immediately implies that

$$\frac{d^2v}{dy^2} = \frac{V}{Y^2}$$

(i.e., without going through the same sequence of arguments that gives us part (a.).)

(c) What do you conclude from part (b.) about $[a]$ if $a$ is ordinary acceleration?

5. In its basic form, work is force times distance, and energy is stored work (and has the same dimension as work). Show that the dimension of energy is equal to $ML^2T^{-2}$ using Newton’s Second Law, $F = ma$.

6. In a diffusion equation for $u = u(x, t)$, it is frequently required that $u_t = \alpha^2 u_{xx}$, with $[u] = ML^{-1}$, $[t] = T$ and $[x] = L$, ($u_t = \partial u/\partial t$, $u_x = \partial u/\partial x$, etc.) In such cases, what can you conclude about $[\alpha]$?

1.2 Scaling to Obtain Dimension-free Models

1.2.1 First order ODE
Let’s begin by extending what we just learned about the logistic model, Equation (1.1). Suppose we set

\[ x = \frac{p}{K} \]  
\[ \tau = rt. \]  

(1.4a)  

(1.4b)

Since \( p \) is a function of \( t \), \( x \) can be considered a function of \( \tau \). In fact if we think of \( p(t) \) as some formula of \( t \), we can substitute \( \tau/r \) for \( t \) in \( p(t) \) and get

\[ x = x(\tau) = \frac{p(\tau/r)}{K} = \frac{p(t)}{K}. \]

We should suspect that \( x(\tau) \) satisfies some differential equation similar to Equation (1.1), and indeed, we can find that

\[
\frac{dx}{d\tau} = \frac{dx}{dp} \frac{dp}{dt} \frac{dt}{d\tau} = \frac{1}{K} \left( rp \left( 1 - \frac{p}{K} \right) \right) \frac{1}{\tau} = \frac{p}{K} \left( 1 - \frac{p}{K} \right) = x(1 - x).
\]

We have actually shown that with \( x \) related to \( p \) and \( \tau \) related to \( t \) as above, if \( p(t) \) solves Equation (1.1), then \( x(\tau) \) solves the similar but streamlined equation

\[
\frac{dx}{d\tau} = x(1 - x). 
\]  

(1.5)

A very similar argument shows the converse, namely, if \( x(\tau) \) satisfies Equation (1.5), if \( t = \tau/r \) and \( p(t) = Kx(\tau) \), then \( p(t) \) solves Equation (1.1).

The scaling in Equations (1.4) was motivated by the desire to reformulate Equation (1.1) in terms of dimensionless variables. (You should verify that \([x] = [\tau] = 1\).) It is not obvious that this can always be done, but the essence of the Buckingham Pi Theorem is that it can always be done for well formulated models, and furthermore that the resulting model has the number of parameters and variables reduced (at least) by the number of dimensions in play in the original model (cf. Logan, [Logan1997]). Such scaling, resulting in dimension-free models, was found in the early 20th century to play important roles, especially in fluid mechanics. For example, it supports important insights into the relationships between flight of model airplanes and flight of the real things. It can simplify work and provide insight into simple models like our logistic one as well.

Let’s see how to interpret solutions to the scaled equation, (1.5), to give information about the original and more complicated one, (1.1). We may interpret \( x \) as giving the proportion of the corresponding \( p \) to its carrying capacity, \( K \), and we can think of \( \tau \) as being (a nondimensional) time, \( t \), scaled in units that correspond very roughly to the doubling time for the population at low levels. (See the concluding discussion of Section 1.) This allows us to solve the simpler equation (with initial conditions formulated...
1. Dimensional Analysis and Scaling

as proportions of $K$ and interpret the simpler solution to give information about the more complicated, original equation. Other benefits will accrue as well.

1.2.2 Second Order ODE

A common second order ODE model is the nonlinear equation for the motion of a pendulum of mass $m$ swinging at the end of a (massless) rod of length $l$ (along a circle of radius $l$). In the presence of damping corresponding to the coefficient $b$ and forcing corresponding to $f(t)$ the equation

$$m \frac{d^2 s}{dt^2} + b \frac{ds}{dt} + k \sin \left( \frac{s}{l} \right) = f(t) \quad (1.6)$$

(along with initial conditions) determines the (signed) distance $s$ along the circle of the mass from vertical equilibrium, at time $t$. It is commonly derived in introductory differential equations texts; see, e.g., ([Borrelli1998]). Let’s see how we can simplify this model by dimensional analysis and scaling.

From the discussion in the last section, it should be clear that $[s] = \text{L}$, $[t] = \text{T}$, $[ds/dt] = \text{LT}^{-1}$.

Since in general the dimension of a derivative follows the rule

$$\left[ \frac{dv}{du} \right] = \left[ \frac{\Delta v}{\Delta u} \right] = \frac{[v]}{[u]}$$

we must have

$$\left[ \frac{d^2 s}{dt^2} \right] = \left[ \frac{d (ds/dt)}{dt} \right] = \left[ \frac{ds}{dt} \right] \frac{\text{LT}^{-1}}{\text{LT}} = \text{LT}^{-2}.$$ 

With that in hand, and assuming that the output of the sine function is dimensionless, you should be able to justify that since $m$ is a mass, i.e., $[m] = \text{M}$, and $l$ is a length, $[l] = \text{L}$, we have

$$[b] = \text{MT}^{-1}, [k] = \text{MLT}^{-2}, [f] = \text{MLT}^{-2}.$$ 

So both $k$ and $f(t)$ are forces (mass times acceleration), consistent with our calling the latter a “forcing” term.

Having obtained all the dimensions, we can scale to nondimensionalize the model. An obvious choice is to set $\theta = s/l$. (Indeed, it is so natural to combine $s/l$ into the angle of displacement $\theta$, that the model (1.6) is frequently derived in terms of $\theta$ instead of $s$.) We also choose $\tau = bt/m$ as a dimensionless time replacement. We will need

$$\frac{d\theta}{d\tau} = \frac{d\theta}{ds} \frac{ds}{dt} \frac{dt}{d\tau} = \frac{1}{l} \frac{ds}{dt} \frac{m}{b}$$
and
\[
\frac{d^2 \theta}{dt^2} = \frac{d}{dt} \left( \frac{d \theta}{dt} \right) = \frac{d}{dt} \left( \frac{m ds}{bl dt} \right) = \frac{m}{bl} \frac{d}{dt} \left( \frac{ds}{dt} \right) = \frac{m d^2 s m}{bl dt^2 b} = \frac{m^2 d^2 s}{b^2 l dt^2}.
\]

It is useful to multiply through (1.6) by \(m/b^2l\) so that it starts off with \(d^2 \theta/dt^2\). Doing this, we find
\[
\frac{m^2 d^2 s}{b^2 l dt^2} + \frac{m ds}{bl dt} + \frac{mk}{b^2 l} \sin(s/l) = \frac{m}{b^2 l} f(t)
\]
or
\[
\frac{d^2 \theta}{dt^2} + \frac{d \theta}{dt} + \omega^2 \sin(\theta) = \gamma(\tau)
\]
with
\[
\omega^2 = \frac{mk}{b^2 l} \quad \text{and} \quad \gamma(\tau) = \frac{m}{b^2 l} f\left(\frac{m\tau}{b}\right).
\]

The result is a dimension free model with 2 fewer parameters.

In summary, when beginning to work with a model, we are well advised to complete a dimensional analysis of it, both to ascertain that it is dimensionally consistent, and to get a feel for some of underlying relationships among variables and parameters. Very frequently it is worth scaling the independent and dependent variables to get dimension-free replacements. There are usually the same numbers of new ones as of old, although sometimes there are fewer dimension-free ones. (The use of “similarity solutions” to diffusion equations is an example here.) Then we are frequently led to natural replacements of parameters, or choose our own. The Buckingham Pi Theorem implies that total number of dimension-free variables and parameters can reduced by the number of dimensions in play in the original model (cf. [Logan1997]).

### 1.2.3 Exercises

1. The classic Lotka-Volterra predator-prey model, as developed in Chapter ??, is

\[
\begin{align*}
\frac{dX}{dt} &= AX - BXY \\
\frac{dY}{dt} &= \varepsilon BXY - DY
\end{align*}
\]

(in which \(\varepsilon B\) is frequently combined into a parameter \(C\)). As is usually the case, there is a variety of useful scalings.
(a) Show that if we set

\[
\begin{align*}
    t &= DT \\
    a &= A/D \\
    x &= B/D \times \\
    y &= B/D \times \\
\end{align*}
\]

then

\[
\begin{align*}
    \frac{dx}{dt} &= ax - xy \\
    \frac{dy}{dt} &= \varepsilon xy - y.
\end{align*}
\]

(b) Show that if we alternatively set

\[
\begin{align*}
    t &= DT \\
    a &= A/D \\
    x &= \varepsilon B/D \times \\
    y &= B/D \times \\
\end{align*}
\]

then

\[
\begin{align*}
    \frac{dx}{dt} &= ax - xy \\
    \frac{dy}{dt} &= xy - y.
\end{align*}
\]

(c) Compare the results of these two scalings in parts (a.) and (b.). Which resulting model is easier to analyze (and why)? Which maintains the sense of nutrient flow in the environment from prey to predator, including loss between the prey and predator populations (due to metabolism, etc.)? Show how the scaling in part (b.) introduces different units of (dimensionless) measurement for the populations. How does that relate to measuring nutrient flow?

2. Consider the model for a “hard,” damped spring,

\[
m \frac{d^2y}{dt^2} + b \frac{dy}{dt} + cy + k^2 y^3 = f(t)
\]

in which \(m\) is a mass, \(y\) is a length of displacement, \(b\) is a positive damping coefficient, \(f(t)\) is a given function and the quantity \(cy + k^2 y^3\) (with \(c, k > 0\)) is the restoring force.
1.2 Scaling to Obtain Dimension-free Models

(a) Assuming that \([t] = T\), use Newton’s second law, \(F = ma\), to conclude that the dimension of the first term, and hence every term, should be a force. Hence, it is consistent to describe \(f(t)\) as a “driving force” and \(cy + k^2 y^3\) as a “restoring force.”

(b) Why do you think the spring is described as being “hard” in comparison with the usual spring model in which \(k = 0\)?

(c) Identify a combination of \(m, b, c, k\) that has length as its dimension.

(d) Identify dimensionless variables \(x\) (to replace \(y\)), and \(\tau\) (to replace \(t\)), a parameter \(\mu\) and function \(\gamma(\tau)\) (both also dimensionless) so that the hard spring model becomes

\[
\mu \frac{d^2 x}{d\tau^2} + \frac{dx}{d\tau} + x + x^3 = \gamma(\tau).
\]

(Any other dimension free, equivalent model might be acceptable. Check with your instructor.)

3. It is remarkable that our scaled pendulum equation (1.7) allows no further choice of the damping coefficient (the coefficient of \(d\theta/d\tau\)). This is a consequence of using the original damping coefficient \(b\) in the scaling of time. Find a different scaling of time to use along with \(\theta = s/l\) that produces a scaled model of the form

\[
\frac{d^2 \theta}{d\tau^2} + \beta \frac{d\theta}{d\tau} + \sin(\theta) = g(\tau).
\]

4. When we consider movement of material or energy, we frequently encounter a diffusion equation for a function, \(u = u(x, t)\), which is a density, e.g., of mass with respect to one or more spatial variables, represented here by the length \(x\). It is frequently found that \(u(x, t)\) satisfies the partial differential equation model \(u_t = u_{xx}\), with \([u] = ML^{-1}, [t] = T\) and \([x] = L\). (\(u_t = \partial u/\partial t, u_x = \partial u/\partial x, \text{etc.}\) See problem 6 of Section 1 regarding \([\alpha]\).)

(a) Just as o.d.e.’s of the form \(dy/dt = f(y)\) are very frequently considered together with an initial condition, \(y(0) = y_0\), so also is the diffusion equation considered with an initial condition of the form \(u(x, 0) = u_0(x)\). Along with \(u_0(x)\), a \textit{fundamental solution} of the diffusion equation,

\[
F(x, t) = \frac{1}{2\alpha \sqrt{\pi t}} \exp\left(-\frac{x^2}{4\alpha^2 t}\right), \quad t > 0,
\]
is used in the expression

\[ u(x,t) = \int_{-\infty}^{\infty} u_0(\xi) F(x - \xi, t) d\xi = \int_{-\infty}^{\infty} \frac{1}{2\alpha \sqrt{\pi t}} \exp \left( \frac{-(x - \xi)^2}{4\alpha^2 t} \right) u_0(\xi) d\xi. \]

for the solution \( u(x,t) \) of both the diffusion equation and the initial condition. (There are technical difficulties with this as \( t \to 0 \), which we will ignore here.)

i. Assuming that the output of the exponential function, \( \exp \), and the number 2 both are dimensionless, find the dimension of the output of the fundamental solution, \([F]\).

ii. What must be \([\xi]\)?

iii. What is \( \left[ \frac{(x-\xi)^2}{4\pi t} \right] \), assuming the 4 is dimensionless?

iv. What must be the dimension, \([u_0(x)]\)?

(b) Verify that \( u \) as given is a solution of the diffusion equation, assuming that you can differentiate through the integral like any regular sum.

(c) For each value of \( t = 1.0, 0.01, 0.0009 \), graph

\[ \frac{1}{\sqrt{\pi t}} \exp \left( \frac{-x^2}{t} \right) \]

as a function of \( x \) (all on the same axes) and compute its integral across the whole \( x \)-axis. If you can, do it by hand; otherwise, use computational assistance.

5. Dimension Matrix: For deeper insight into the processes of dimensional analysis and scaling it is helpful to construct a dimension matrix for a system.

(a) We consider the logistic model (1.1) as an example. Consider the independent set of dimensions \( \mathbf{M}, \mathbf{T} \). We construct the dimension matrix \( D \) with a row corresponding to each of the independent dimensions and a column corresponding to each variable or parameter in the given model. The columns contain the exponents for the dimensions. Thus for the logistic model (1.1) we obtain \( D \) as the \( 2 \times 4 \) array in the lower right of

\[
\begin{pmatrix}
    p & t & r & K \\
    1 & 0 & 0 & 1 \\
    0 & 1 & -1 & 0
\end{pmatrix}
\]

Fill in the details of this argument for the logistic model as follows:
1.2 Scaling to Obtain Dimension-free Models

i. Any candidate for a dimension-free quantity of the form
\[ q = p^{\alpha_1} t^{\alpha_2} r^{\alpha_3} K^{\alpha_4} \]
will have dimension
\[ [q] = [p]^{\alpha_1} [t]^{\alpha_2} [r]^{\alpha_3} [K]^{\alpha_4} = M^{\alpha_1} + \alpha_4 T^{\alpha_2 - \alpha_3}. \]

ii. In order for \( q \) to be dimension-free, it is necessary and sufficient that the exponent vector \( \alpha = [\alpha_1, \cdot \cdot \cdot, \alpha_4]^T \) satisfy
\[ \alpha_1 + \alpha_4 = 0 \]
\[ \alpha_2 - \alpha_3 = 0; \]
or equivalently,
\[ D \alpha = 0, \]
that is, \( \alpha \) must be in \( N(D) \), the null-space of \( D \).

iii. Hence, in this particular example we can expect to find a 2-dimensional set of possible choices for \( \alpha \).

iv. What choices of \( \alpha \) did we use for our dimension-free model, (1.5)?

(b) Consider the pendulum model (1.6), including the force \( f \) as a “parameter”:

i. Identify the independent set of dimensions used in the discussion of section (1.2.2) above.

ii. Find the corresponding dimension matrix, \( D \).

iii. Write down the form of a candidate for a dimension-free quantity \( q \) analogous to the form in part (a.i.) and identify a set of linear equations for the vector \( \alpha \) of exponents to guarantee that \( q \) is dimension-free, as in part (a.ii.).

iv. What is the dimension of the set of all \( \alpha \) that give dimension-free equivalent models?

v. What choices of \( \alpha \) did we use for our dimension-free model, (1.7)?

(c) In the general case of a dimensionally consistent model stated in terms of \( n \) variables and parameters involving \( k \) independent dimensions, sketch an argument to show that we can expect to find an equivalent model stated in terms of a total of \( n - k \) dimension-free variables and parameters.
1. Dimensional Analysis and Scaling
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